STRONG DITKIN ALGEBRAS WITHOUT BOUNDED RELATIVE UNITS

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Abstract. In a previous note the author gave an example of a strong Ditkin algebra which does not have bounded relative units in the sense of Dales. In this note we investigate a certain family of Banach function algebras on the one point compactification of $\mathbb{N}$, and see that within this family are many easier examples of strong Ditkin algebras without bounded relative units in the sense of Dales.

1. INTRODUCTION

Regularity conditions for Banach function algebras have important applications in several areas of functional analysis, including automatic continuity theory and the theory of Wedderburn decompositions (see, for example, [**3**]). There is also a close connection between regularity and the theory of decomposable operators, as was shown by Neumann in [**6**]. It is thus important both to investigate the connections between these regularity conditions and other conditions that a Banach function algebra may satisfy, and also to find out what the relationships are between these regularity conditions. For a survey of many regularity conditions and their relationships to each other, see [**5**].

The aim of this note is to investigate some of the stronger regularity conditions for a particular class of Banach function algebras, following on from the work in [**4**]. In [**4**], the author solved a problem posed by Bade (see [**2**]), by showing that every strong Ditkin algebra has bounded relative units at each point of its character space. The proof was very short and elementary. In fact, unknown to the author, essentially the same argument had been used earlier by Bachelis, Parker and Ross ([**1**]) to prove a special case of the same result. (I am grateful to Jan Stegeman for later pointing out the existence of this paper to me). Also in [**4**], the author constructed an example of a strong Ditkin algebra which does not satisfy the stronger condition of having bounded relative units in the sense of Dales (see below). In Section 2 we shall introduce a particular class of Banach function algebras on $\mathbb{N}_\infty$, and determine precisely which combinations of regularity properties are possible for algebras in this class. In particular we shall show that this class includes many further examples of strong Ditkin algebras which do not have bounded relative units in the sense of Dales. These examples are easier than the example constructed in [**4**], and show that interesting combinations of properties are possible even when working with Banach function algebras on $\mathbb{N}_\infty$. First we recall the standard notation and definitions which we shall need.

NOTATION. For any compact, Hausdorff space X we denote the algebra of all continuous, complex-valued functions on X by $C(X)$. Let A be a commutative, unital algebra. We denote by Φ_A the character space of A. Let S, T be subsets of A. We denote by $S \cdot T$ the set

$$\{ab : a \in S, b \in T\}.$$

We denote the one point compactification of $\mathbb{N}$ by $\mathbb{N}_\infty$ (so that $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$). For $f \in C(\mathbb{N}_\infty)$, $\|f\|_\infty$ denotes the usual uniform norm of f on $\mathbb{N}_\infty$, so that

$$\|f\|_\infty = \sup\{|f(n)| : n \in \mathbb{N}\}.$$

DEFINITION 1.1: Let X be a compact space. A *function algebra* on X is a subalgebra of $C(X)$ which contains all of the constant functions and which separates the points of X. A *Banach function algebra* on X is a function algebra on X with a complete algebra norm.

Throughout we shall consider only unital Banach algebras. Using the Gelfand transform, every commutative, semisimple Banach algebra may be regarded as a Banach function algebra on its character space.

NOTATION. Let A be a (unital) Banach function algebra on Φ_A and let E be a closed subset of Φ_A. We define two ideals $J(E)$, $I(E)$ in A as follows:

$$J(E) = \{f \in A : f \text{ vanishes on some neighbourhood of } E\};$$

$$I(E) = \{f \in A : f(E) \subseteq \{0\}\}.$$

For $\phi \in \Phi_A$, we set $M_\phi = I(\{\phi\})$ and we denote $J(\{\phi\})$ by J_ϕ.

DEFINITION 1.2: Let A be a Banach function algebra on Φ_A and let $\phi \in \Phi_A$. Then A is *strongly regular at* ϕ if J_ϕ is dense in M_ϕ; A satisfies *Ditkin's condition at* ϕ if $J_\phi \cdot M_\phi$ is dense in M_ϕ. We say that A has *bounded relative units at* ϕ if there exists $C \geq 1$ such that, for all compact sets $E \subseteq \Phi_A \backslash \{\phi\}$, there is $f \in J_\phi$ with $\|f\| \leq C$ such that $f(E) \subseteq \{1\}$. The algebra A is *strongly regular* if it is strongly regular at every point of Φ_A; A is a *Ditkin algebra* if it satisfies Ditkin's condition at every point of Φ_A; A is a *strong Ditkin algebra* if every maximal ideal of A has a bounded approximate identity and A is strongly regular. The Banach function algebra A *has spectral synthesis* if, for every closed set $E \subseteq \Phi_A$, $J(E)$ is dense in $M(E)$.

Clearly every Ditkin algebra is strongly regular. In the special case where $\Phi_A = \mathbb{N}_\infty$ it is easy to see that if A is a Ditkin algebra, then A has spectral synthesis (this is not true in general).

There are two definitions for a Banach function algebra A to have bounded relative units. One is that, for every $\phi \in \Phi_A$, A has bounded relative units at ϕ. If this condition holds we shall say that A has *bounded relative units in the sense of Bade*. The other definition, used more frequently, is stronger, insisting that the constant C involved does not depend on ϕ. If this stronger condition holds we shall say that A has *bounded relative units in the sense of Dales*. For uniform algebras, these two conditions are equivalent. But they differ for general Banach function algebras, as was shown in [**4**].

The following result combines some standard theory with [**4**, Theorem 5].

PROPOSITION 1.3. *Let A be a Banach function algebra. Then A is a strong Ditkin algebra if and only if A is strongly regular and has bounded relative units in the sense of Bade.*

As we mentioned earlier, in [**4**] there is an example of a strong Ditkin algebra which does not have bounded relative units in the sense of Dales. In the next section we shall give some easier examples of this.

2. BANACH FUNCTION ALGEBRAS ON $\mathbb{N}_\infty$.

We shall now introduce the class of Banach function algebras that we shall work with. From now on we shall use α to denote a sequence of positive real numbers, with $\alpha = (\alpha_n)_{n=1}^\infty$. Given such a sequence α, we define A_α by

$$A_\alpha = \left\{ f \in C(\mathbb{N}_\infty) : \sum_{n=1}^\infty \alpha_n |f(n+1) - f(n)| < \infty \right\}.$$

It is easy to see that A_α is a subalgebra of $C(\mathbb{N}_\infty)$, and that A_α is a Banach function algebra, where the norm of a function $f \in A_\alpha$ is given by

$$\|f\| = \|f\|_\infty + \sum_{n=1}^\infty \alpha_n |f(n+1) - f(n)|.$$

It is also easy to see that the character space of A_α is just $\mathbb{N}_\infty$ (this follows from [**4**, Proposition 6], for example).

We shall now investigate the further properties of these algebras. We shall see that A_α is always a Ditkin algebra (and hence strongly regular). In fact we will see that A_α has a stronger property: for all $x \in \mathbb{N}_\infty$, M_x has an approximate identity (not necessarily bounded) consisting of a sequence of elements of J_x. Of course, for each $x \in \mathbb{N}$, it is clear that $J_x = M_x$, and that these ideals have an identity. So we only need to check what happens at the point ∞. We begin by looking at the particular sequence of functions $(e_k) \subseteq A_\alpha$ defined by

$$e_k(n) = \begin{cases} 1 & \text{if } n \le k, \\ 0 & \text{otherwise.} \end{cases}$$

We shall see that this sequence of functions in J_∞ always has a subsequence which is an approximate identity for M_∞, although the sequence (e_k) need not itself be such an approximate identity. First we need an elementary lemma.

LEMMA 2.1. *Let $(n_k)_{k=1}^\infty$ be a strictly increasing sequence of natural numbers. Then the following conditions are equivalent:*
(a) (e_{n_k}) is an approximate identity for M_∞;
(b) for all $f \in M_\infty$, $\lim_{k\to\infty} \alpha_{n_k} f(n_k + 1) = 0$;
(c) for all $f \in M_\infty$, $\lim_{k\to\infty} \alpha_{n_k} f(n_k) = 0$.

PROOF: For any $f \in A_\alpha$ we know that $\sum_{k=1}^\infty \alpha_n |f(n+1) - f(n)| < \infty$, and so $\lim_{n\to\infty} \alpha_n |f(n+1) - f(n)| = 0$. From this it is clear that (b) and (c) are equivalent.

Now let $f \in M_\infty$. Then

$$(f - e_{n_k} f)(j) = \begin{cases} 0 & \text{if } j \le n_k, \\ f(j) & \text{otherwise.} \end{cases}$$

We have that

$$\|f - e_{n_k} f\| = \|f - e_{n_k} f\|_\infty + \sum_{j=n_k+1}^{\infty} (\alpha_j |f(j+1) - f(j)|) + \alpha_{n_k} |f(n_k+1)|. \tag{2.1}$$

Since the first two terms on the right hand side of (2.1) tend to zero as $k \to \infty$, it is clear that $\lim_{k\to\infty} \|f - e_{n_k} f\| = 0$ if and only if $\lim_{k\to\infty} \alpha_{n_k} f(n_k+1) = 0$. It is now immediate that (a) and (b) are equivalent, as required. ■

The sequence (e_k) is not always an approximate identity for M_∞. For example, suppose that α is the sequence (α_n) defined by

$$\alpha_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Then (e_k) is not an approximate identity for M_∞. This can be seen by considering the function $f \in C(\mathbb{N}_\infty)$ which satisfies

$$f(j) = 2^{-k}$$

whenever $j, k \in \mathbb{N}$ are such that $2^{k-1} \le j < 2^k$.

When $j = 2^k - 1$, $\alpha_j |f(j+1) - f(j)| = 2^{-k-1}$. All other $|f(j+1) - f(j)|$ are 0, so that $f \in A_\alpha$, and $f \in M_\infty$. But $\alpha_{2^k} f(2^k) = \frac{1}{4}$ for all k, and so $\alpha_n f(n)$ does not tend to 0 as $n \to \infty$. Thus, by Lemma 2.1, (e_k) is not an approximate identity for M_∞.

We now continue our investigation into when (e_{n_k}) is an approximate identity.

THEOREM 2.2. *Let (n_k) be as in Lemma 2.1.*

(a) Suppose that, for all $k \in \mathbb{N}$, $\alpha_{n_k} = \inf\{\alpha_j : j \ge n_k\}$. Then (e_{n_k}) is an approximate identity for M_∞.

(b) The sequence (e_{n_k}) is a bounded approximate identity for M_∞ if and only if the sequence (α_{n_k}) is bounded.

PROOF: To prove (a), suppose that (α_{n_k}) satisfies the given condition. Let $f \in M_\infty$. Then we have, for $k \in \mathbb{N}$,

$$f(n_k) = \sum_{j=n_k}^{\infty} (f(j) - f(j+1)).$$

Thus

$$\begin{aligned} |\alpha_{n_k} f(n_k)| &\le \sum_{j=n_k}^{\infty} \alpha_{n_k} |f(j) - f(j+1)| \\ &\le \sum_{j=n_k}^{\infty} \alpha_j |f(j) - f(j+1)|. \end{aligned}$$

Since the last sum tends to 0 as $k \to \infty$, it follows that $\lim_{k\to\infty} \alpha_{n_k} f(n_k) = 0$. It now follows from Lemma 2.1 that (e_{n_k}) is an approximate identity for M_∞.

To prove (b), first note that, for all k,

$$\|e_{n_k}\| = 1 + \alpha_{n_k}.$$

Thus it is clear that the sequence (e_{n_k}) is bounded in norm if and only if (α_{n_k}) is bounded. It remains to show that, in this case, (e_{n_k}) is also an approximate identity for M_∞. But this is immediate from Lemma 2.1. The result follows. ∎

We can now see that (e_k) is itself often an approximate identity.

COROLLARY 2.3. *If the sequence α is either nondecreasing or bounded then (e_k) is an approximate identity for M_∞.*

PROOF: This is immediate from Theorem 2.2. ∎

We can now prove our main result about the existence of approximate identities in M_∞.

THEOREM 2.4. *The sequence (e_k) always has a subsequence which is an approximate identity in M_∞ in A_α.*

PROOF: If (α_n) has a bounded subsequence, then the result is immediate from Theorem 2.2 (b). Thus we may assume that (α_n) diverges to ∞. But then it is easy to choose (n_k) such that the conditions of Theorem 2.2. (a) are satisfied, and so (e_{n_k}) is an approximate identity for M_∞. ∎

As a corollary, we obtain immediately the fact that each A_α is a Ditkin algebra.

COROLLARY 2.5. *For every sequence of positive real numbers α, the Banach function algebra A_α is a Ditkin algebra.*

PROOF: For $x \in \mathbb{N}$ it is clear that $J_x = M_x$ and that these ideals have an identity. It remains to check that $J_\infty \cdot M_\infty$ is dense in M_∞, but this is immediate from Theorem 2.4. The result follows. ∎

Of course this means that A_α is always strongly regular. By the nature of the algebra, it follows that A_α is always separable. Since $\Phi_{A_\alpha} = \mathbb{N}_\infty$, it also follows that A_α has spectral synthesis.

We now characterise those sequences α for which the algebra A_α has various regularity properties.

THEOREM 2.6. *Let $\alpha = (\alpha_n)_{n=1}^\infty$ be any sequence of positive real numbers. Then*

(a) A_α is a strong Ditkin algebra if and only if $\liminf_{n\to\infty} \alpha_n < \infty$, i.e. (α_n) does not diverge to ∞.

(b) In A_α, M_∞ has a bounded approximate identity if and only if $\liminf_{n\to\infty} \alpha_n < \infty$.

(c) *The Banach function algebra A_α has bounded relative units in the sense of Bade if and only if* $\liminf_{n\to\infty} \alpha_n < \infty$.

(d) *The algebra A has bounded relative units in the sense of Dales if and only if the sequence (α_n) is bounded.*

PROOF: We know that A_α is a Ditkin algebra (and hence strongly regular). Also, for all $x \in \mathbb{N}$, $J_x = M_x$, and M_x has an identity. Thus, by Proposition 1.3, it is clear that (a), (b), (c) are equivalent, and that for each of these we only need to check the relevant condition at the point ∞. Suppose first that $\liminf_{n\to\infty} \alpha_n < \infty$. Then (α_n) has a bounded subsequence, and so it follows from from Theorem 2.2 (b) that M_∞ has a bounded approximate identity. Thus A_α is a strong Ditkin algebra, and has bounded relative units in the sense of Bade.

Conversely, suppose that A_α has bounded relative units in the sense of Bade. Then there is a norm bounded sequence of functions $(f_n) \subseteq J_\infty$ such that, for all $n \in N$, $f_n(\{n\}) = \{1\}$. Set $M = \sup_n \|f_n\|$. We show that $\liminf_{n\to\infty} \alpha_n \le M$. Suppose, for contradiction, that $\liminf_{n\to\infty} \alpha_n > M$. Choose $N \in \mathbb{N}$ such that $\inf\{\alpha_j : j \ge N\} > M$. Then we have

$$\begin{aligned} 1 &= |f_N(N)| \\ &= \left| \sum_{j=N}^{\infty} (f_N(j) - f_N(j+1)) \right| \\ &\le \sum_{j=N}^{\infty} |(f_N(j) - f_N(j+1))| \\ &\le \frac{\|f_N\|}{\inf\{\alpha_j : j \ge N\}} < 1. \end{aligned}$$

This contradiction proves that, as claimed, $\liminf_{n\to\infty} \alpha_n \le M < \infty$. Parts (a) to (c) now follow.

To see (d), suppose first that (α_n) is bounded. Set $M = \sup_n \alpha_n$. For $n \in \mathbb{N}$, the norm of the identity of M_n is at most $2M+1$. Also, with e_n as above, $\|e_n\| \le M+1$, and so A_α has bounded relative units in the sense of Dales, with bound $2M+1$. Conversely, since the norm of the identity in M_n is at least α_n, it is immediate that if A_α has bounded relative units in the sense of Dales then (α_n) must be bounded. The result follows. ■

This result can be used to give many elementary examples of strong Ditkin algebras which do not have bounded relative units in the sense of Dales.

COROLLARY 2.7. *Let α be any sequence of positive real numbers such that α is unbounded but does not diverge to ∞. Then A_α is a strong Ditkin algebra which does not have bounded relative units in the sense of Dales.*

PROOF: This is immediate from Theorem 2.6. ■

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